ON LINEAR SYSTEMS AND A CONJECTURE OF D. C. BUTLER

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Abstract. Let C be a smooth irreducible projective curve of genus g and L a line bundle of degree d generated by a linear subspace V of $H^0(L)$ of dimension $n+1$. We obtain new results on the stability of the kernel of the evaluation map.

1. Introduction

Let C be a smooth irreducible projective curve of genus g. Let (L,V) be a linear series of type $(d,n+1)$ with L a generated line bundle of degree d on C and $V \subset H^0(L)$ a linear subspace of dimension $n+1$ which generates L. We suppose that $n \geq 2$. The kernel of the evaluation map $V \otimes \mathcal{O} \to L$, denoted by $M_{V,L}$, fits in an exact sequence

$$0 \to M_{V,L} \to V \otimes \mathcal{O} \to L \to 0. \qquad (1.1)$$

The vector bundle $M_{V,L}$, and its dual $M^*_{V,L}$, have been studied from various points of view. In particular, note that the sequence (1.1) can also be seen as the pullback of the (dual of the) Euler sequence via the morphism $\phi_L : C \to \mathbb{P}^n(V^*) = \mathbb{P}^n$ defined by the sections of L, so that $M_{V,L} \simeq \phi_L^*(\Omega_{\mathbb{P}^n}(1))$. When $V = H^0(L)$, the bundle $M_{H^0(L),L}$ is often denoted by M_L and can be seen as an integral transform of L (see, for example, [9]).

One of the main questions about $M_{V,L}$ is when it is stable. It is well known that, if $g \geq 1$ and $d > 2g$, then M_L is stable [9] (see also [7, Theorem 1.2]); this has been extended to some cases where V has low codimension in [12]. It is also known that M_L is semistable for C a general curve of genus $g \geq 2$ [14]. In [8], D. C. Butler made a conjecture [8, Conjecture 2] for the more general case where L is replaced by a semistable vector bundle E and V is any linear subspace that generates E; for the case of a line bundle, this can be stated in the following form (except that Butler restricts to the case $g \geq 3$).

Conjecture 1.1. *For C a general curve of genus $g \geq 1$ and a general choice of (L,V), the bundle $M_{V,L}$ is semistable.*

There are many variants of Conjecture 1.1, of which we consider the following two (see section 2 for the definition of a *Petri* curve).

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Conjecture 1.2. *For C a general curve of genus $g \geq 3$ and a general choice of (L, V), the bundle $M_{V,L}$ is stable.*

Conjecture 1.3. *For C a Petri curve of genus $g \geq 3$ and a general choice of (L, V), the bundle $M_{V,L}$ is stable.*

Conjectures 1.2 and 1.3 can fail for $g \leq 2$. For $g = 0, 1$, this is obvious; if $g = 0$, there are no stable bundles if rank $n \geq 2$, while, for $g = 1$, the same is true if $\gcd(n, d) > 1$. For $g = 2$, see [11, Théorème 2] and [3, Theorem 8.2]. For $g \geq 3$, a sharp lower bound for the existence of (L, V) of type $(d, n + 1)$ on a Petri curve is $d \geq g + n - \left\lfloor \frac{g}{n+1} \right\rfloor$. Our aim is to describe for which values of (g, d, n), $M_{V,L}$ is stable or semistable.

Much work has been done on these conjectures, although none of them has been completely solved. Techniques used include the use of deformations, classical Brill-Noether theory and coherent systems (see section 2 for details of some of the results obtained). Our object in this paper is to put together the most far reaching of these results and to extend the technique used in our previous paper [3] to prove the above conjectures in more cases.

In section 2, we recall some definitions and state some known results. Section 3 contains our main results. Using some results from [3], we prove the following theorem which extends the known results on Conjecture 1.1.

Theorem 3.3. *Suppose that $g = ns + t \geq 1$ with $s \geq 0$ and $0 \leq t \leq n - 1$. Then Conjecture 1.1 holds for all $d \geq g + n - \left\lfloor \frac{g}{n+1} \right\rfloor$ in the following cases:*

(i) $s = 0$;
(ii) $t \leq 3$;
(iii) $t \geq 4$ *and* $s \geq \min\{n - t - 2, t - 3\}$;
(iv) $t \geq 5$, $s = t - 4$ *and* $n \leq 4t - 7$.

Also in section 3, we obtain improved versions of certain propositions in [3], which allow us to extend this theorem further.

Theorem 3.10. *Suppose $g = ns + t$ with $s \geq 1$ and $0 \leq t \leq n - 1$. If either (a) $t \geq n - 5$ or (b) $t \leq 7$ and $n \leq 16$, then Conjecture 1.1 holds for all $d \geq g + n - \left\lfloor \frac{g}{n+1} \right\rfloor$.*

It follows in particular that Conjecture 1.1 holds for all $d \geq g + n - \left\lfloor \frac{g}{n+1} \right\rfloor$ when either $n \leq 13$ or $g \leq 21$ (see Corollaries 3.12 and 3.13).

In section 4, we obtain some consequences of the lemmas in section 3 for Conjectures 1.3 and 1.2. Recall that, in [3], we proved Conjecture 1.3 for all possible (g, d, n) with $n \leq 4$ as well as many other cases. Now we present a short list of possible exceptions for $n = 5$, which enables us to prove Conjecture 1.2 completely in this case (Theorem 4.1). For $n = 6$, we obtain slightly longer lists of possible exceptions for $n = 6$ (Theorem 4.3). Finally, we prove

Theorem 4.5. *Suppose that $n \geq 7$ and that either $n \equiv 0 \mod 3$ and $g \geq \frac{n^2-2n+3}{3}$ or $n \not\equiv 0 \mod 3$ and $g \geq \frac{n^2-2n}{3}$. Then Conjecture 1.3 holds for all $d \geq g+n-\left\lfloor \frac{g}{n+1} \right\rfloor$.*

This is a substantial improvement on the previously known result that Conjecture 1.3 holds for $g \geq n^2-1$.

2. Background

We suppose always that C is a smooth projective curve of genus $g \geq 1$ defined over the complex numbers with canonical bundle K. When we say that C is *general*, we mean that C lies in some unspecified non-empty Zariski open subset of the moduli space of curves of genus g. The curve C is a *Petri* curve if the multiplication map

$$H^0(L) \otimes H^0(K \otimes L^*) \to H^0(K)$$

is injective for every line bundle L on C. It is a standard fact that Petri curves do define a non-empty Zariski-open subset of the moduli space, so any result which is valid for Petri curves of genus g is also valid for the general curve of genus g.

Remark 2.1. It is known that, on a Petri curve, a linear series (L, V) of type (g, d, n) exists if and only if

$$d \geq g+n-\left\lfloor \frac{g}{n+1} \right\rfloor.$$

Moreover, Conjecture 1.3 holds whenever

$$g+n-\left\lfloor \frac{g}{n+1} \right\rfloor \leq d \leq g+n \tag{2.1}$$

and also for any $d \geq g+n-\left\lfloor \frac{g}{n+1} \right\rfloor$ when $g \geq n^2-1$ (see [8, 6]).

For later use, we recall some facts about coherent systems (see [4] for more details). A coherent system (E, V) of type (r, d, k) on C is a pair consisting of a vector bundle E of rank r and degree d and a linear subspace $V \subset H^0(E)$ of dimension k. The coherent system (E, V) is said to be *generated* if the evaluation map $V \otimes \mathcal{O} \to E$ is surjective. There is a concept of stability dependent on a real parameter α and there exist moduli spaces $G(\alpha; r, d, k)$ of α-stable coherent systems of type (r, d, k); a necessary condition for the non-emptiness of $G(\alpha; r, d, k)$ is that $\alpha > 0$. There are finitely many critical values $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_L$ of α; as α varies, the concept of α-stability remains constant between two consecutive critical values. We denote by $G_0(r, d, k)$ (resp. $G_L(r, d, k)$) the moduli spaces corresponding to $0 < \alpha < \alpha_1$ (resp. $\alpha > \alpha_L$). It is well known that, if $(E, V) \in G_0(r, d, k)$, then E is semistable; moreover, if E is stable, then, for any linear subspace V of dimension k, $(E, V) \in G_0(r, d, k)$. We write also

$$\begin{aligned} U(r,d,k) &:= \{(E,V) \in G_L(r,d,k) | E \text{ is stable}\} \\ &= \{(E,V) | E \text{ is stable and } (E,V) \in G(\alpha; r,d,k) \text{ for all } \alpha > 0\}. \end{aligned}$$

For any generated coherent system (E, V), we have an exact sequence, analogous to (1.1):

$$0 \to M_{V,E} \to V \otimes \mathcal{O} \to E \to 0. \tag{2.2}$$

Suppose that $h^0(E^*) = 0$. Dualising (2.2), we obtain a new generated coherent system $D(E, V) := (M^*_{V,E}, V^*)$, called the *dual span* of (E, V). The original form of Butler's conjecture is as follows.

Conjecture 2.2. [8, Conjecture 2] *Let C be a general curve of genus $g \geq 3$ and let $(E, V) \in G_0(r, d, r + n)$ with (E, V) generated. Then $D(E, V) \in G_0(n, d, r + n)$.*

Remark 2.3. The case $r = 1$ of Conjecture 2.2 is equivalent to Conjecture 1.1. In fact (L, V) is α-stable for all $\alpha > 0$. If $M_{V,L}$ is semistable (hence also $M^*_{V,L}$), then it is easy to see, using the fact that $D(L, V)$ is generated and $h^0(M_{V,L}) = 0$, that $D(L, V) \in G_0(n, d, n + 1)$. The converse is clear.

Remark 2.4. Following on from Remark 2.3, we note that, if $M_{V,L}$ is stable, then $D(L, V) \in U(n, d, n + 1)$. When the space of linear systems (L, V) is irreducible (which is not always the case), Conjecture 1.2 is therefore equivalent to the assertion that, under the stated conditions, there exists $(E, V) \in U(n, d, n + 1)$ with (E, V) generated. For a Petri curve, one can be more precise and say that Conjecture 1.3 is equivalent to the assertion that $U(n, d, n + 1) \neq \emptyset$ [3, Proposition 9.6]. From this and [5, Theorem 5.4] (see also [10, 11]), it follows that, when $g < n$, Conjecture 1.3 holds also when

$$g + n \leq d \leq 2n. \tag{2.3}$$

Lemma 2.5. *Suppose that $g \geq 3$ and there exists a generated linear system (L, V) of type $(d, n + 1)$ on C such that $M_{V,L}$ is stable. Then Conjecture 1.3 holds for linear systems of type $(d + an, n + 1)$ on C and a a positive integer.*

Proof. This follows from Remark 2.4 and [3, Remark 2.2]. □

Corollary 2.6. *Suppose that C is a Petri curve of genus $g \geq 3$, $n \geq 1$ and Conjecture 1.3 holds for all d with*

$$g + n - \left\lfloor \frac{g}{n+1} \right\rfloor \leq d < g + n + b,$$

where $b + \left\lfloor \frac{g}{n+1} \right\rfloor > n - 1$. Then Conjecture 1.3 holds for all $d \geq g + n - \left\lfloor \frac{g}{n+1} \right\rfloor$.

Proof. This follows at once from the lemma. □

We turn now to known results on Conjectures 1.1 and 1.2. A recent preprint [1] addresses Conjecture 1.1.

Proposition 2.7. [1, Theorem 1.8] *Let C be a general curve of genus $g \geq 1$ and (L, V) a general linear series of type $(d, n + 1)$. Suppose $g = ns + t$ with $0 \leq t \leq n - 1$. If*

$$d \geq \max\{g + n + \min\{n - t, t - 2\}, g + n\}, \tag{2.4}$$

then $M_{V,L}$ is semistable.

For $g = 1, 2$, this is already known for any smooth curve and the bound given by (2.4) is sharp [3, Theorems 8.1 and 8.2].

Corollary 2.8. *Suppose $g = ns + t$ with $s \geq 0$ and either $0 \leq t \leq \min\{3, n-1\}$ or $t = n-1$. Then Conjecture 1.1 holds for all $d \geq g + n - \left\lfloor \frac{g}{n+1} \right\rfloor$.*

Proof. This follows at once from Proposition 2.7 and Remark 2.1. □

An earlier paper [2] addresses Conjectures 1.1 and 1.2.

Proposition 2.9. [2, Proposition 1.6, Theorem 1.7 and Remark 1.10] *Let C be a general curve of genus $g \geq 2$ and (L, V) a general linear series of type $(d, n+1)$ with $n \geq 3$. Then*

- *if $u = \left\lceil \frac{g-1}{n+2} \right\rceil$ and $d \geq n(u+1) + 1$, then $M_{V,L}$ is semistable;*
- *if $u = \left\lceil \frac{g-2}{n+2} \right\rceil$ and $d \geq \max\{n(u+2) + 1, 3n + 2\}$, then $M_{V,L}$ is stable.*

Note that, if $n \geq 3$, the second case of Proposition 2.9 applies to all $d \geq g + n + 1$ whenever $g \geq n^2$ (which is slightly stronger than the condition $g \geq n^2 - 1$ mentioned above).

Teixidor [15] has also some results on Conjecture 1.2. All of these results are based on deforming reducible nodal curves and so do not specify the meaning of the term "general curve". On the other hand, our results in [3] assume only that C is a Petri curve and address Conjecture 1.3. To extend the range of values for which Conjecture 1.3 is known, we used wall-crossing formulae for coherent systems. In the next section, we will exploit these techniques further and obtain improved results on Conjectures 1.3 and 1.1.

3. Main results

We begin with a proposition which improves some of the results in [3, Section 6].

Proposition 3.1. *Suppose that C is a Petri curve of genus g, $3 \leq g \leq (2n-1)(n-2)$ and $n \geq 4$. If (L, V) is a general linear series of type (d, n) with*

$$g + n - \left\lfloor \frac{g}{n+1} \right\rfloor \leq d < g + n + \frac{1}{n-2}\left(2g - n\left\lfloor \frac{g}{n-1} \right\rfloor\right), \tag{3.1}$$

then $M_{V,L}$ is stable.

Proof. From Remark 2.1, we have only to consider $d > g + n$. For this, one needs to check the proofs of [3, Propositions 6.9 and 6.12]. For [3, Proposition 6.9], it is clear that (3.1) is sufficient. For [3, Propopsition 6.12], one must note that the hypothesis on g implies that $\frac{1}{n}\left(g + n + \frac{1}{n-2}\left(2g - n\left\lfloor \frac{g}{n-1} \right\rfloor\right)\right) \leq \frac{2g}{2n-1} + 2$. One needs also to check that the first inequality in [3, (6.7)] is sufficient to give a contradiction. Finally one must show that a corrected version of [3, (6.8)] implies [3, (6.6)]. Our result then follows from [3, Propositions 6.9, 6.10 and 6.12] □

Corollary 3.2. *Let C be a Petri curve of genus $g \geq 3$ and $n \geq 4$. Suppose further that $g \geq \frac{n^2-n}{2}$ if n is odd or $g \geq \frac{n^2-n+2}{2}$ if n is even. Then Conjecture 1.3 holds for all $d \geq g+n-\frac{g}{n+1}$.*

Proof. We know this is true if $g \geq n^2-1$. If $g < n^2-1$, then the hypothesis on g in the proposition holds, so, by (2.1) and (3.1), Conjecture 1.3 holds for

$$g+n-\left[\frac{g}{n+1}\right] \leq d < g+n+\frac{1}{n-2}\left(2g-n\left\lfloor\frac{g}{n-1}\right\rfloor\right).$$

By Corollary 2.6, it is therefore sufficient to prove that

$$\frac{1}{n-2}\left(2g-n\left\lfloor\frac{g}{n-1}\right\rfloor\right)+\left\lfloor\frac{g}{n+1}\right\rfloor > n-1.$$

One can check that this holds under the stated restrictions on g. □

This is already an improvement on the known bound $g \geq n^2-1$, which we will improve further in section 4. For $n=4$, we know that, in fact, Conjecture 1.3 is always true [3, Theorem 7.3].

Theorem 3.3. *Suppose that $g = ns+t \geq 1$ with $n \geq 3$, $s \geq 0$ and $0 \leq t \leq n-1$. Then Conjecture 1.1 holds for all $d \geq g+n-\left\lfloor\frac{g}{n+1}\right\rfloor$ in the following cases:*

(i) $s = 0$;
(ii) $t \leq 3$;
(iii) $t \geq 4$ *and* $s \geq \min\{n-t-2, t-3\}$;
(iv) $t \geq 5$, $s = t-4$ *and* $n \leq 4t-7$.

Proof. (i) If $s=0$, $g<n$, so Conjecture 1.1 holds for $g+n \leq d \leq 2n$ by [10, 11] (see also [5, Theorem 5.4]). By Proposition 2.7, the conjecture holds also when $d \geq g+n+n-t = 2n$ and hence for any $d \geq g+n-\left\lfloor\frac{g}{n+1}\right\rfloor$.

(ii) See Corollary 2.8.

(iii) We can suppose $s \geq 1$. The result will follow from Propositions 2.7 and 3.1 and Corollary 3.2 if we can prove

$$\frac{1}{n-2}\left(2g-n\left\lfloor\frac{g}{n-1}\right\rfloor\right) > \min\{n-t-1, t-3\}.$$

Substituting $g = ns+t$, this is equivalent to proving

$$ns+2t-n\left\lfloor\frac{s+t}{n-1}\right\rfloor > (n-2)\min\{n-t-1, t-3\}. \tag{3.2}$$

Note that, for fixed n and t, the left hand side of (3.2) is an increasing function of s, so we can assume $s = \min\{n-t-2, t-3\}$.

Suppose first that $s = n-t-2 < t-3$. Then it is sufficient to show that

$$n(n-t-2)+2t > (n-2)(n-t-1),$$

which reduces to $n > 2$.

Suppose now that $t-3 \le n-t-2$ and $s = t-3$. Then it is sufficient to prove

$$n(t-3) + 2t > (n-2)(t-3),$$

which is true.

(iv) We again have to prove (3.2). By (iii), we can assume that $t-4 < n-t-2$. So (3.2) will hold if

$$n(t-4) + 2t > (n-2)(t-3),$$

i.e. $n \le 4t-7$. □

Corollary 3.4. *If $n \le 9$, then Conjecture 1.1 holds for all $d \ge g+n-\left\lfloor \frac{g}{n+1} \right\rfloor$.*

Proof. By (ii), (iii) and (iv), Conjecture 1.1 holds for $s \ge \min\{n-t-2, t-4\}$, hence for all $s \ge 1$ when $n \le 9$. The result now follows from (i). □

In order to progress further by the methods of [3], we need to improve the results of [3, Section 6]. We begin by recalling some notation.

Let C be a Petri curve and let (E, W) be a generated coherent system of type $(n, d, n+1)$ with $H^0(E^*) = 0$. Then $(E, W) \in G_L(n, d, n+1)$. If E is not stable, there exists an extension of coherent systems

$$0 \to (E_1, V_1) \to (E, W) \to (E_2, V_2) \to 0 \tag{3.3}$$

(see [3, (6.1)]) with (E_i, V_i) of types (n_i, d_i, k_i) and

$$k_2 \ge n_2 + 1, \quad 1 + \frac{1}{n_2}\left(g - \left\lfloor \frac{g}{n_2+1} \right\rfloor\right) \le \mu(E_2) \le \mu(E). \tag{3.4}$$

Moreover (E_2, V_2) is generated and $h^0(E_2^*) = 0$.

The following two lemmas are similar to [3, Propositions 6.9 and 6.12].

Lemma 3.5. *Suppose that C is a Petri curve, $n \ge 4$,*

$$d < g + n + \frac{1}{n-3}\left(3g - n\left\lfloor \frac{g}{n-2} \right\rfloor\right) \tag{3.5}$$

and $n_2 \le n-3$. Then no extension (3.3) *exists satisfying* (3.4).

Proof. If such an extension exists, then

$$1 + \frac{1}{n-3}\left(g - \left\lfloor \frac{g}{n-2} \right\rfloor\right) \le \frac{d}{n}$$

(compare the proof of [3, Proposition 6.9]). This contradicts the hypothesis on d. □

Lemma 3.6. *Suppose that C is a Petri curve, $n \ge 3$ and*

$$d < g + n + \min\left\{\frac{(n^2-n-2)g}{2(n-1)^2}, n + \frac{g}{2n-1}\right\}. \tag{3.6}$$

Then there exists no extension (3.3) *satisfying* (3.4) *with $k_2 \ge n_2 + 2$.*

Proof. Suppose that an extension (3.3) with the stated properties exists. Since (E_2, V_2) is generated, we can choose a subspace W_2 of $H^0(E_2)$ of dimension $n_2 + 2$ which generates E_2. This yields an exact sequence

$$0 \to N_2 \to W_2 \otimes \mathcal{O} \to E_2 \to 0,$$

where N_2 has rank 2 and $h^0(N_2) = 0$. Dualising this sequence and using the fact that $h^0(E_2^*) = 0$, we have $h^0(N_2^*) \geq n_2 + 2$; moreover $\deg N_2^* = d_2$. Following the proof of [3, Proposition 6.12], we consider three possibilities.

(i) $h^0(L_1) \leq 1$ for every line subbundle L_1 of N_2^*.

(ii) There exists an exact sequence

$$0 \to L_1 \to N_2^* \to L_2 \to 0$$

with $h^0(L_2) = s, 2 \leq s \leq n_2$.

(iii) There exists an exact sequence as in (ii) with $h^0(L_2) \geq n_2 + 1$, $h^0(L_1) \geq 2$.

In case (i), [13, Lemma 3.9] implies that $h^0(\det N_2^*) \geq 2n_2 + 1$, so, by Remark 2.1,

$$d_2 \geq g + 2n_2 - \frac{g}{2n_2 + 1}. \tag{3.7}$$

In case (ii),

$$d_2 = \deg L_1 + \deg L_2 \geq g + s - 1 - \frac{g}{s} + g + n_2 + 1 - s - \frac{g}{n_2 + 2 - s}.$$

The right hand side of this expression takes its minimum value at $s = 2$ (and $s = n_2$), so

$$d_2 \geq 2g + n_2 - \frac{(n_2 + 2)g}{2n_2}. \tag{3.8}$$

Finally, in case (iii),

$$d_2 = \deg L_1 + \deg L_2 \geq g + 1 - \frac{g}{2} + g + n_2 - \frac{g}{n_2 + 1} > 2g + n_2 - \frac{(n_2 + 2)g}{2n_2}.$$

So (3.8) holds again in this case.

Now recall from (3.4) that $\frac{d}{n} \geq \frac{d_2}{n_2}$; moreover $n_2 \leq n - 1$. Substituting in (3.7) and (3.8), we obtain a contradiction to (3.6). $\square$

Remark 3.7. The second term in the minimum of (3.6) is essentially irrelevant for our purposes since $n + \frac{g}{2n-1} > n - 1$ (see Corollary 2.6).

For convenience, we state also

Lemma 3.8. *Suppose that C is a Petri curve, $n \geq 2$ and $d > g + n$. Then the extensions (3.3) satisfying (3.4) with $n_2 = n - 1$, $k_2 = n$ depend on at most $\beta(n, d, n + 1) - 1$ parameters.*

Proof. This is [3, Proposition 6.10]. $\square$

Our final lemma is a version of Lemma 3.8 for $n_2 = n - 2$.

Lemma 3.9. *Suppose that C is a Petri curve, $g \geq 5$, $n \geq 5$ and $d > g + n$. Then the extensions* (3.3) *satisfying* (3.4) *with $n_2 = n - 2$, $k_2 = n - 1$ depend on at most $\beta(n, d, n+1) - 1$ parameters.*

Proof. Since (E_2, V_2) is generated and $h^0(E_2^*) = 0$, we have $(E_2, V_2) \in G_L(n-2, d_2, n-1)$, so (E_2, V_2) depends on $\beta(n-2, d_2, n-1)$ parameters. It is therefore sufficient to prove the following two statements:

(a) (E_1, V_1) depends on at most $\beta(2, d_1, 2) + 2 = 2d_1 - 1$ parameters;
(b) $C_{12} > h^0(E_1^* \otimes N_2 \otimes K) + 2$.

(Here,

$$C_{12} = (k_1 - n_1)(d_2 - n_2(g-1)) + n_2 d_1 - k_1 k_2 = (n-2)d_1 - 2(n-1) \tag{3.9}$$

and N_2 is the kernel of the evaluation map $V_2 \otimes \mathcal{O} \to E_2$ (see [4, (19), (9) and (11)]).)

Note first that (E_1, V_1) is generically generated, for otherwise there would exist a line subbundle L_1 of E_1 with $V_1 \subset H^0(L_1)$, contradicting the fact that $(E, W) \in G_L(n, d, n+1)$. It follows that there is an exact sequence

$$0 \to (L_1, W_1) \to (E_1, V_1) \to (L_2, W_2) \to 0, \tag{3.10}$$

where (L_1, W_1) and (L_2, W_2) are linear systems of type $(e_1, 1)$ and $(e_2, 1)$ respectively. Note that $e_i \geq 0$ for $i = 1, 2$ and $e_1 + e_2 = d_1$. To prove (a), observe that (E_1, V_1) depends on at most $e_1 + e_2 + \dim \mathrm{Ext}^1((L_2, W_2), (L_1, W_1)) - 1$ parameters. Moreover, using [4, (11)], we have

$$\dim \mathrm{Hom}((L_2, W_2), (L_1, W_1)) \leq \max\{1, e_1 - e_2\}, \quad \mathrm{Ext}^2((L_2, W_2), (L_1, W_1)) = 0.$$

So, using [4, (8) and (9)],

$$\dim \mathrm{Ext}^1((L_2, W_2), (L_1, W_1)) \leq \max\{1, e_1 - e_2\} + e_2 - 1 = \max\{e_2, e_1 - 1\} \leq d_1.$$

This proves (a).

For (b), note that, by (3.9) and (3.10), it is sufficient to prove that

$$h^0(L_1^* \otimes N_2 \otimes K) + h^0(L_2^* \otimes N_2 \otimes K) < (n-2)d_1 - 2n. \tag{3.11}$$

We have

$$\deg(L_i^* \otimes N_2 \otimes K) = 2g - 2 - e_i - d_2. \tag{3.12}$$

We consider four possibilities

(i) $e_i + d_2 \leq 2g - 2$ for $i = 1, 2$. In this case, Clifford's Theorem and (3.12) give

$$h^0(L_1^* \otimes N_2 \otimes K) + h^0(L_2^* \otimes N_2 \otimes K) \leq 2g - d_2 - \frac{1}{2}(e_1 + e_2) = 2g - d_2 - \frac{d_1}{2}.$$

Substituting $d_1 = d - d_2$ here and in (3.11) and rearranging, we see that it is sufficient to prove that

$$\frac{(2n-3)d}{2} - \frac{(2n-5)d_2}{2} > 2g + 2n.$$

Now, by (3.4), $d_2 \leq \frac{(n-2)d}{n}$, so this inequality will hold if

$$\frac{d}{2n}(n(2n-3)-(n-2)(2n-5)) > 2(g+n),$$

i.e.

$$d > \frac{2n}{3n-5}(g+n).$$

This holds since $d > g+n$ and $n \geq 5$.

(ii) $e_i + d_2 > 2g-2$ for $i = 1,2$. In this case, by (3.12), we need only to prove that $(n-2)d_1 > 2n$. Since $d_1 \geq \frac{2d}{n}$ and $d > g+n$, it is enough to show that $g \geq \frac{2n}{n-2}$, which holds for $n \geq 5$, $g \geq 4$.

(iii) $e_1 + d_2 \leq 2g-2 < e_2 + d_2$. In this case, Clifford's Theorem and (3.12) give

$$h^0(L_1^* \otimes N_2 \otimes K) + h^0(L_2^* \otimes N_2 \otimes K) \leq g - \frac{1}{2}(e_1 + d_2).$$

By (3.11) and using the fact that $e_1 \geq 0$ and $d_1 = d - d_2$, we see that it is enough to prove that

$$(n-2)d - \frac{(2n-5)d_2}{2} > g + 2n.$$

Now use $d_2 \leq \frac{(n-2)d}{n}$ and rearrange; the assertion will then follow if we can prove that $\frac{d}{2n}(5n-10) > g+2n$. Since $d > g+n$ with $g \geq 5$, $n \geq 5$, this holds.

(iv) $e_2 + d_2 \leq 2g-2 < e_1 + d_2$. This case proceeds exactly as for (iii) with e_1 replaced by e_2. □

Theorem 3.10. *Suppose that $g = ns + t$ with $s \geq 1$ and $0 \leq t \leq n-1$. If either (a) $t \geq n-5$ or (b) $t \leq 7$ and $n \leq 16$, then Conjecture 1.1 holds for all $d \geq g + n - \left\lfloor \frac{g}{n+1} \right\rfloor$.*

Proof. In view of Theorem 3.3, we can assume that $5 \leq t \leq n-4$, so in particular $n \geq 9$. This implies that

$$\frac{n^2-n-2}{2(n-1)^2} \geq \frac{3}{n-3}.$$

It follows from (2.1), Proposition 2.7 and Lemmas 3.5 - 3.9 that it is sufficent to prove that

$$\frac{1}{n-3}\left(3g - n\left\lfloor \frac{g}{n-2} \right\rfloor\right) > \min\{n-t-1, t-3\}. \tag{3.13}$$

Substituting $g = ns + t$, we see that (3.13) is equivalent to

$$2ns + 3t - n\left\lfloor \frac{2s+t}{n-2} \right\rfloor > (n-3)\min\{n-t-1, t-3\}.$$

The left hand side of this inequality is an increasing function of s, so we need only prove it for the case $s = 1$, i.e. we must prove that one of the following two inequalities holds.

$$2n + 3t - n\left\lfloor \frac{t+2}{n-2} \right\rfloor > (n-3)(n-t-1); \tag{3.14}$$

$$2n + 3t - n\left\lfloor \frac{t+2}{n-2} \right\rfloor > (n-3)(t-3). \tag{3.15}$$

For $t \geq n-5$, we rewrite (3.14) as

$$2n + nt - n\left\lfloor \frac{t+2}{n-2} \right\rfloor > (n-3)(n-1).$$

The left hand side of this inequality is an increasing function of t, so, for (a), we can assume $t = n-5$. In this case, the inequality holds for $n \geq 4$. This completes the proof for $t \geq n-5$.

For $t \leq 7$, we rewrite (3.15) as

$$5n - 9 > (n-6)t + n\left\lfloor \frac{t+2}{n-2} \right\rfloor.$$

Since we are assuming that $n \geq 9$, the right hand side is an increasing function of t, so it is sufficient to prove the inequality for $t = 7$, in which case it holds for $n \leq 16$, $n \neq 11$. For $n = 11$ and $t \leq 6$, this argument still works. If $n = 11$ and $t = 7$, we use case (a). □

Remark 3.11. For $t = 6$, the argument above still works for $n \leq 26$. For $t = 5$, it works without restriction on n.

Corollary 3.12. *If $n \leq 13$, then Conjecture 1.1 holds for all $d \geq g + n - \left\lfloor \frac{g}{n+1} \right\rfloor$.*

Proof. This follows from Theorem 3.3(i) and Theorem 3.10. □

Corollary 3.13. *If $g \leq 21$, then Conjecture 1.1 holds for all (n, d) with $d \geq g+n-\left\lfloor \frac{g}{n+1} \right\rfloor$.*

Proof. By Theorem 3.3(i) and Corollary 3.12, we can assume $s \geq 1$ and $n \geq 14$. The result now follows from Theorem 3.10(b). □

Remark 3.14. The first case for which the arguments above do not apply is $(n, g, d) = (14, 22, 41)$. For this and other cases, we would need a further improvement of the lemmas. In fact, Lemma 3.6 works in this case and the analogue of Lemma 3.5 for $n_2 \leq n-4$ also works, so we need a version of Lemma 3.9 for $n_2 = n-3$, at least in this special case.

4. Conjectures 1.2 and 1.3

We know from [3] that Conjecture 1.3 holds for $n \leq 4$. In this section we consider larger values of n. We obtain also results on Conjecture 1.2.

Theorem 4.1. *Suppose that $n = 5$ $g \geq 3$ and $d \geq g + 5 - \left\lfloor \frac{g}{6} \right\rfloor$. Then*

- *Conjecture 1.3 holds except possibly in the following cases:*
  $g = 3, d = 12$
  $g = 4, d = 12, 13, 17, 18$
  $g = 5, d = 13, 18.$
- *Conjecture 1.2 holds in all cases.*

Proof. Suppose first that C is a Petri curve. By Proposition 3.1 and Lemmas 3.5 - 3.9, Conjecture 1.3 holds for

$$g+5-\left\lfloor\frac{g}{6}\right\rfloor \le d < g+5+\min\left\{\frac{1}{2}\left(3g-5\left\lfloor\frac{g}{3}\right\rfloor\right), \frac{9g}{16}, 5+\frac{g}{9}\right\}. \tag{4.1}$$

If

$$\min\left\{\frac{1}{2}\left(3g-5\left\lfloor\frac{g}{3}\right\rfloor\right), \frac{9g}{16}\right\}+\left\lfloor\frac{g}{6}\right\rfloor > 4, \tag{4.2}$$

then, by Corollary 2.6 and Remark 3.7, Conjecture 1.3 holds for all $d \ge g+n-\left\lfloor\frac{g}{6}\right\rfloor$. One can check that (4.2) holds for $g \ge 6$. For $g \le 5$, we check cases, using (4.1), Lemma 2.5 and [3, Propositions 6.6 and 6.8].

In all the excluded cases, we have $\gcd(5,d)=1$. Conjecture 1.2 therefore follows from Proposition 2.7 or Proposition 2.9. □

Remark 4.2. By Lemma 2.5, to complete the proof of Conjecture 1.3 for $n=5$, it is sufficient to prove the following cases: $g=3, d=12$; $g=4, d=12, 13$; $g=5, d=13$.

Theorem 4.3. *Suppose that $n=6$, $g \ge 3$ and $d \ge g+6-\left\lfloor\frac{g}{7}\right\rfloor$. Then*

- *Conjecture 1.3 holds except possibly in the following cases*
  $g=3, d=14$
  $g=4, d=14, 15, 20, 21$
  $g=5, d=14, 15, 16, 20, 21, 22$
  $g=6, d=16, 22, 28$
  $g=8, d=18, 24, 30$
- *Conjecture 1.2 holds whenever Conjecture 1.3 holds and in addition when $d \ge 20$.*

Proof. For Conjecture 1.3, we use the same argument as in the proof of Theorem 4.1. The range of values to be considered is now

$$g+6-\left\lfloor\frac{g}{7}\right\rfloor \le d < g+6+\min\left\{g-2\left\lfloor\frac{g}{4}\right\rfloor, \frac{14g}{25}, 6+\frac{g}{11}\right\}. \tag{4.3}$$

For $g \ge 9$, we can check that

$$\min\left\{g-2\left\lfloor\frac{g}{4}\right\rfloor, \frac{14g}{25}\right\}+\left\lfloor\frac{g}{7}\right\rfloor > 5,$$

and it follows from Corollary 2.6 and Remark 3.7 that Conjecture 1.3 holds. For lower values of g , we argue case by case. The fact that there are no excluded cases for $g=7$ depends on [3, Proposition 6.8] as well as (4.3).

The result for Conjecture 1.2 follows from Proposition 2.9. □

Remark 4.4. To complete the proof of Conjecture 1.3 for $n=6$, it is sufficient to prove the cases $g=3, d=14$; $g=4, d=14, 15$; $g=5, d=14, 15, 16$; $g=6, d=16$; $g=8, d=18$.

Theorem 4.5. *Suppose that $n \geq 7$ and that either $n \equiv 0 \mod 3$ and $g \geq \frac{n^2-2n+3}{3}$ or $n \not\equiv 0 \mod 3$ and $g \geq \frac{n^2-2n}{3}$. Then Conjecture 1.3 holds for all $d \geq g + n - \lfloor \frac{g}{n+1} \rfloor$.*

Proof. Note first that

$$\frac{(n^2-n-2)g}{2(n-1)^2} \geq \frac{1}{n-3}\left(3g - n\left\lfloor \frac{g}{n-2}\right\rfloor\right).$$

For $n \geq 9$, this has already been noted in the proof of Theorem 3.10; for $n = 7$ or $n = 8$, it holds for $g \geq n-2$. In view of Lemmas 3.5 - 3.9 and Corollary 2.6, it is therefore sufficient to prove that

$$\frac{1}{n-3}\left(3g - n\left\lfloor \frac{g}{n-2}\right\rfloor\right) + \left\lfloor \frac{g}{n+1}\right\rfloor > n-1.$$

Note that, if this is true for $g = c(n-2)$ with c an integer, then it is true for all larger g. The proof is completed by identifying the minimal value of c and then considering the range of values $(c-1)(n-2) < g < c(n-2)$. □

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